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CASSON'S INVARIANT AND QUADRATIC RECIPROCITY

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IT IS A corollary of Casson's work on the λ-invariant, that if M is a homology 3-sphere and every irreducible, 2-dimensional unitary representation of $\pi_1(M)$ is non-degenerate (i.e. has no infinitesimal deformations), then the number of such representations is even [1, 2]. Our main result shows that this property is not special to 3-manifold groups:

THEOREM 1. *Let Γ be a finitely-presented, perfect group and suppose there are only finitely many inequivalent, non-trivial representations of Γ in $SU(2)$. Then the number of such representations is even.*

Remark. Since the space of representations is compact, one can replace the finiteness condition by the requirement that each representation ρ be isolated; but this is still weaker than Casson's condition, that ρ be non-degenerate. Curiously, if Γ is a 3-manifold group some of whose representations in $SU(2)$ are isolated but *degenerate*, Casson's work implies a result which differs from Theorem 1 and asserts that the sum of certain multiplicities is even. The authors are not at present aware of a 3-manifold group for which these multiplicities are other than ± 1.

Theorem 1 will be deduced from a central result in the theory of quaternion algebras, closely related to quadratic reciprocity, and attributable to Hilbert and Hasse [5, 4]. Let A be a quaternion algebra (a 4-dimensional, central simple algebra) over a global field k, and for each place v of k, let $A_v = A \otimes_k k_v$ be the corresponding algebra over the local field k_v. Each A_v is either a division algebra (which is then uniquely determined up to isomorphism) or the matrix algebra $M_2(k_v)$. In the former case, one says that A is ramified at v, in the latter, that A is unramified. Hasse's form of Hilbert's reciprocity law asserts that the number of places at which ramification occurs is finite and even. Since there are no non-commutative division algebras over $\mathbb{C}$, this reduces to a statement about the real and finite places of k.

Now let $\sigma:\Gamma \to SU(2)$ be a non-trivial representation. With σ we associate the field k generated by the traces,

$$k = \mathbb{Q}(tr \circ \sigma(\gamma))_{\gamma \in \Gamma},$$

which we consider as an abstract field endowed with a canonical embedding $\iota: k \hookrightarrow \mathbb{R}$. We write T for the trace-function $tr \circ \sigma$ regarded as a map $T:\Gamma \to k$, being careful to distinguish it from the composite $\iota \circ T$. In this situation there is a naturally given k-algebra, namely the $\iota(k)$-linear span of the matrices $\sigma(\gamma)$ in $M_2(\mathbb{C})$. Rather than take this as a definition however,

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we give a construction (cf. [3]) for an isomorphic algebra A which depends only on T and not otherwise on σ. Consider the group algebra $R = k[\Gamma]$ and the k-algebra homomorphism

$$\hat{\sigma}: R \to M_2(\mathbb{C})$$

which extends σ. Since Γ is perfect, the image of $\hat{\sigma}$ is non-abelian, and its complex-linear span is therefore all of $M_2(\mathbb{C})$. Since $tr(ab)$ is a non-degenerate form on $M_2(\mathbb{C})$, it now follows that the kernel of $\hat{\sigma}$ is the two-sided ideal

$$I = \{r \in R \mid T(r_1 r r_2) = 0 \quad \forall r_1, r_2 \in R\}.$$

(Here we have extended T linearly as a map $R \to k$.) We define

$$A = A_T = R/I.$$

The construction makes clear that A is isomorphic to the image of $\hat{\sigma}$, and also shows that $A_{\iota T} = A \otimes_{k,\iota} \mathbb{R}$ is isomorphic to the $\mathbb{R}$-linear span of $\sigma(\Gamma)$, which is $\mathbb{H}$, the Hamiltonian quaternions. From this it follows that A is a (non-trivial) quaternion algebra.

PROPOSITION 1. *k is a number field.*

Proof. Because Γ is finitely generated, k is a finitely-generated extension of $\mathbb{Q}$: indeed, it is a subfield of the larger field generated by the matrix elements of all the $\sigma(\gamma)$, which in turn is generated by the matrix elements of any set of generators. It remains to show that $tr(\sigma(\gamma))$ is algebraic. Since $SU(2)$ may be regarded as the set of real points of an algebraic group G defined over $\mathbb{Q}$, the homomorphisms $\rho: \Gamma \to SU(2)$ comprise the real points, $X(\mathbb{R})$, for a variety $X = \mathrm{Hom}(\Gamma, G)$, also defined over $\mathbb{Q}$. Because of our hypothesis that the representations fall into finitely many conjugacy classes, the proposition will follow from the next lemma, applied to the function $f: X(\mathbb{R}) \to \mathbb{R}$ given by $f(\rho) = tr(\rho(\gamma))$.

LEMMA. *Let $X/\mathbb{Q}$ be an algebraic variety, and f a $\mathbb{Q}$-rational function on X. If $x_0 \in X(\mathbb{R})$ and $f(x_0) \notin \bar{\mathbb{Q}}$, then there exists a continuous curve on $X(\mathbb{R})$, passing through x_0, on which f is non-constant.*

Proof. We present the function field, F, of X as the quotient field of the ring

$$\mathbb{Q}[y_1, \ldots, y_n]/I.$$

It is sufficient to treat the case of an affine variety, so we may suppose that the functions y_i are regular and embed $X(\mathbb{R})$ in $\mathbb{R}^n$. As $f(x_0) \notin \bar{\mathbb{Q}}$, the field

$$F_{x_0} = \mathbb{Q}(y_1(x_0), \ldots, y_n(x_0))$$

is transcendental over $\mathbb{Q}$. Choose $f_1, \ldots, f_m \in F$, with $f = f_1$, such that $\{f_i(x_0)\}$ forms a transcendency basis for F_{x_0}. Each $y_i(x_0)$ satisfies an irreducible polynomial equation, $P_i(y_i(x_0)) = 0$, where

$$P_i \in \mathbb{Q}(f_1(x_0), \ldots, f_m(x_0))[z] \subset \mathbb{R}[z].$$

As the P_i are irreducible, they have simple roots, it follows that the real roots of P_i are real, single-valued algebraic functions of the coefficients in some open neighborhood of P_i in real coefficient space. In particular, all the y_i are real-valued functions in some neighborhood, U, of $u_0 = (f_1(x_0), \ldots, f_m(x_0))$ in $\mathbb{R}^m$. These functions give a map $\varphi: U \to \mathbb{R}^n$ carrying u_0 to $x_0 \in X(\mathbb{R}) \subset \mathbb{R}^n$. Since the coordinates of u_0 are independent transcendental elements and

φ is algebraic, the image of φ lies entirely in $X(\mathbb{R})$. The path $\zeta(t) = \varphi(f_1(x_0) + t, \ldots, f_m(x_0))$ therefore has the required property.

PROPOSITION 2. *If Γ is perfect, A is unramified at every finite place.*

Proof. From the construction, Γ is naturally represented in A^1, the group of elements of A of reduced norm 1, and therefore also in A_v^1 for each place v. If A_v is a division algebra over a non-Archimedean local field k_v, the valuation on k_v extends uniquely to a discrete valuation on A_v (see [4]). Let $\mathscr{P}_v$ be the set of elements of A_v of positive valuation, the maximal ideal in the ring, $\mathcal{O}_v$, of elements of non-negative valuation. Then A_v^1 is contained in $\mathcal{O}_v^\times$ which has a Hausdorff filtration

$$\mathcal{O}_v^\times \supset (1 + \mathscr{P}_v)^\times \supset (1 + \mathscr{P}_v^2)^\times \supset \ldots$$

with abelian quotients. It follow that the image of Γ in A_v^1, and hence the image of Γ in A, is the identity, contrary to hypothesis.

To summarize, we have associated with each representation σ a triple (k, T, ι) consisting of a number field k, a trace-function $T: \Gamma \to k$ and an embedding $\iota: k \hookrightarrow \mathbb{R}$. These have the property that the associated k-algebra $A = A_T$ is a quaternion algebra, unramified at all finite places; and $A \otimes_{k,\iota} \mathbb{R} \cong \mathbb{H}$, which is to say that A is ramified at the real place ι. The triple determines σ up to conjugacy, for σ is the composite of the representation $\Gamma \to A_\iota^1$ and an isomorphism $A_\iota^1 \cong SU(2)$. If $v: k \to \mathbb{R}$ is now another embedding at which A is ramified, we obtain a homomorphism $\Gamma \to A_v^1 \cong \mathbb{H}^1 = SU(2)$, with trace T. This proves:

PROPOSITION 3. *The conjugacy classes of homomorphisms $\rho: \Gamma \to SU(2)$ with trace-field k and trace-function T are in one-to-one correspondence with the real places of k at which A_T ramifies.*

Proof of Theorem 1. If we classify the non-trivial representations σ according to their trace-fields and trace-functions, then by Proposition 3 the number of representations in the class corresponding to (k, T) is equal to the number of real places of ramification for A_T. When combined with Proposition 2, Hilbert's reciprocity law implies that the number of such places is even. So the number of non-trivial representations is even also.

Finally, it is natural to ask what becomes of Theorem 1 without the hypothesis that Γ is perfect. In general there is little one can say, but we do have the following result for groups whose derived series stabilizes:

THEOREM 2. *Let Γ be a finitely-presented group whose derived series $\Gamma \supseteq \Gamma' \supseteq \Gamma^{(2)} \supseteq \ldots$ eventually stabilizes, and suppose there are only finitely many inequivalent representations of Γ in $SU(2)$. Then the parity of the number of such representations is determined by $\Gamma/\Gamma^{(3)}$.*

LEMMA. *If Γ is a finitely-presented group which admits only finitely many representations in $SU(2)$, of which $\rho: \Gamma \to SU(2)$ is one, then $\rho(\Gamma)$ is either finite or dense in $SU(2)$.*

Proof of Lemma. The only proper closed subgroups of $SU(2)$ are finite groups, the Cartan subgroup $T = e^{2\pi i\mathbb{R}}$, and its normalizer $N(T) = T \cup \varepsilon T$ (where $\varepsilon^2 = -1$ and $\varepsilon^{-1} t \varepsilon = t^{-1}$ for $t \in T$). If the closure of $\rho(\Gamma)$ were T then we would obtain infinitely many inequivalent representations by composing ρ with the nth-power homomorphism $p(n): T \to T$. If n is odd, $p(n)$ extends to an endomorphism of $N(T)$, so the same argument rules out the possibility that $N(T)$ is the closure of $\rho(\Gamma)$.

Proof of Theorem 2. Suppose $\Gamma^{(n)} = \Gamma^{(n+1)}$, so that $\Gamma^{(n)}$ is perfect, and let $\Delta = \Gamma/\Gamma^{(n)}$ be the solvable quotient. We divide the representations of Γ into two classes: those whose kernel contains $\Gamma^{(n)}$ (and whose image is therefore solvable), and the remainder. If σ belongs to the latter class then the associated field k and quaternion algebra A may be defined as before; indeed, the construction of A depends only on the image of Γ being non-abelian. Furthermore, under the same hypothesis on σ, the conclusion of Proposition 2 still holds: since the image of Γ contains a perfect subgroup, the same proof by contradiction applies. Thus the proof of Theorem 1 adapts readily to showing that the representations σ whose kernels do not contain $\Gamma^{(n)}$ are even in number. It is therefore only necessary to consider the representations ρ in the other class—those which factor through Δ. In considering these we can abandon our previous framework entirely and argue ad hoc. For any such ρ, the closure of $\rho(\Delta)$ is, like Δ, solvable and hence, by the Lemma, finite. The only solvable finite subgroups of $SU(2)$ are the cyclic groups and the binary dihedral, tetrahedral and octahedral groups. The binary octahedral group, $\mathbb{O}$, has an outer automorphism of order 2 which is not obtained by conjugating by an element of $SU(2)$, so equivalence classes of homomorphisms $\Delta \to \mathbb{O}$ occur in pairs. If H is cyclic, binary dihedral, or binary tetrahedral, $H^{(3)}$ is trivial, so homomorphisms $\Delta \twoheadrightarrow H$ correspond to homomorphisms $\Delta/\Delta^{(3)} \twoheadrightarrow H$, and their number is determined by $\Delta/\Delta^{(3)} = \Gamma/\Gamma^{(3)}$.

REFERENCES

1. S. Akbulut and J. McCarthy: *Casson's invariant for oriented homology three-spheres*. Princeton University Press, to appear.
2. M. F. Atiyah: New invariants of three and four dimensional manifolds. In *The mathematical heritage of Hermann Weyl*, Proc. Symp. Pure Math. **48**, R. Wells, ed. American Mathematical Society, RI, (1988).
3. H. Bass: Groups of integral representation type. *Pacific J. Math.*, **86** (1980), 15–51.
4. M.-F. Vignéras: *Arithmétique des algèbres de quaternions*. Lecture Notes in Mathematics, **800**, Springer-Verlag, (1980).
5. A. Weil: *Basic Number Theory*. Springer-Verlag, (1974).

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